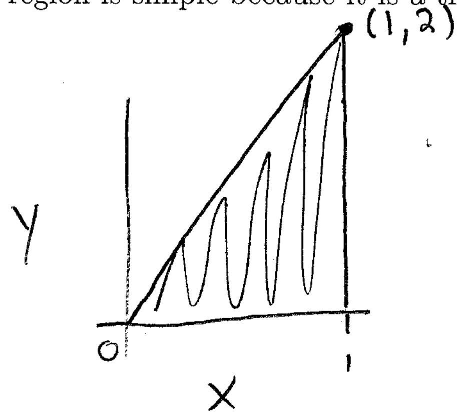
Fundamental Theorem of Calculus

April 4, 2013

What is the area under the curve $y = 2x$ from $x = 0$ to $x = 1$? Calculating the area of this region is simple because it is a triangle.

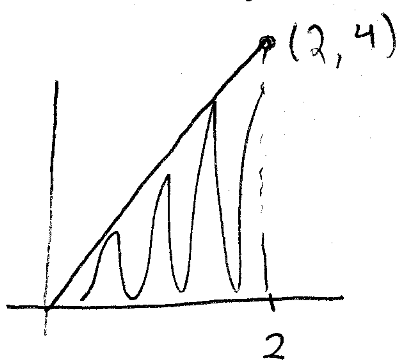


$$A = \frac{1}{2} bh$$

$$= \frac{1}{2} 1 \cdot 2 = \frac{1}{2} \cdot 2 = 1$$

Can you find the area under the curve $y = 2x$ from $x = 0$ to $x = \ldots$

0	0
1	1
2	4
3	9
4	16
5	25
a	a^2

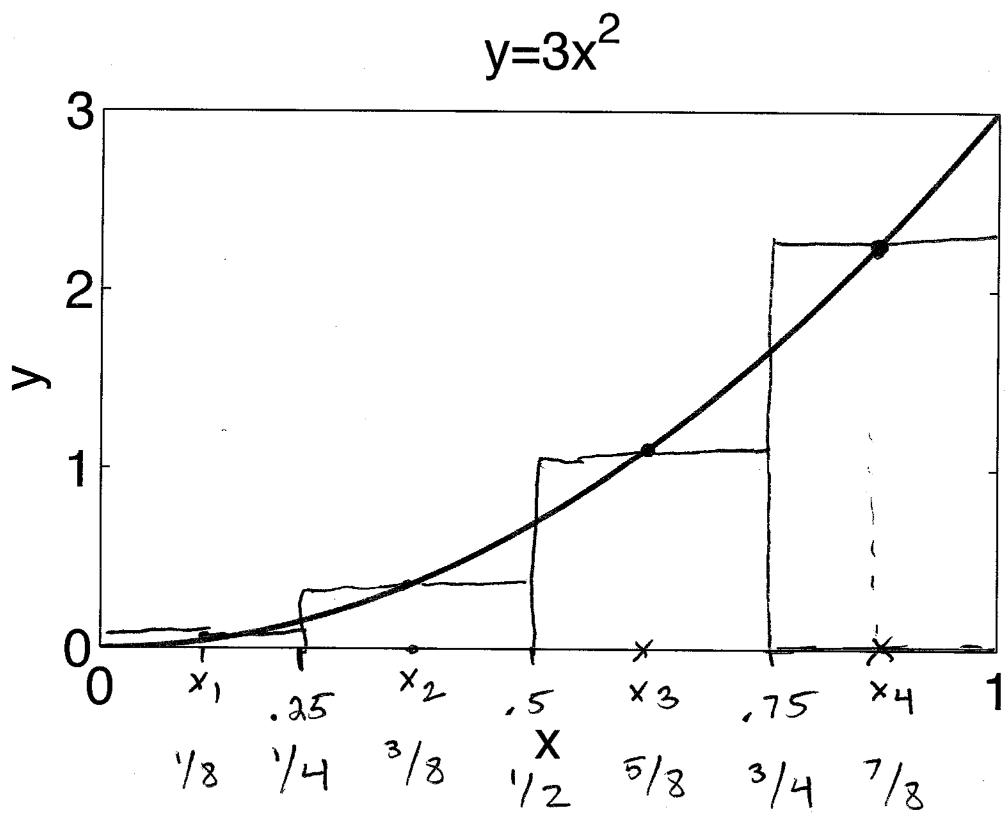


$$A = \frac{1}{2} bh = \frac{1}{\cancel{2}} \cancel{2} 4 = 4$$

What is the area under the curve $y = 3x^2$ from $x = 0$ to $x = 1$? Unlike $y = 2x$, this isn't a simple region that we can find the area of using geometry. However, we can *approximate* this area by constructing a bunch of rectangles whose total area is approximately the same as the area under the curve. This is called a **Riemann Sum**. Let's do this to approximate the area under $y = 3x^2$ from $x = 0$ to $x = 1$.

Step 1: Divide the interval $[0, 1]$ into equally sized *sub*-intervals.

Step 2: In each of these parts, draw a rectangle whose height is equal to the height of the graph of y in the middle of the subinterval (**Note:** You don't have to pick the middle point, you could pick the right endpoint, the left endpoint, or any other point in the subinterval you want).



Let's call x_1 the point in the middle of the 1st subinterval, x_2 the point in the middle of the 2nd subinterval, and so on.

Area of Rectangle j = $\Delta x \cdot y(x_j)$

Δx = rectangle width

j	1	2	3	4
x_j	1/8	3/8	5/8	7/8
$f(x_j)$	.047	.422	1.17	2.30
Area of Rectangle j	.012	.106	.29	.58

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 1$ (4 rectangles, midpoint rule): .988

The above estimate is not exact. How could we do better?

j	1	2	3	4	5	6	7	8
x_j	1/ 16	3/ 16	5/ 16	7/ 16	9/ 16	11/ 16	13/16	15/16
$f(x_j)$	.0117	.1055	.2930	.5742	.949	1.418	1.980	2.637
Area of Rectangle j	.001	.013	.037	.072	.119	.177	.248	.330

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 1$ (8 rectangles, midpoint rule): .997

If we continued to take more and more rectangles, what do you think our area estimates would get closer and closer to? 1

Now, let's use the same procedure to approximate the area under $y = 3x^2$ from $x = 0$ to $x = 2$ using 2 rectangles:

j	1	2
x_j	1/2	3/2
$f(x_j)$	.75	6.75
Area of Rectangle j	.75	6.75

$A \approx 7.5$

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 2$ (2 rectangles, midpoint rule):

Now use 4 rectangles to approximate the area under $y = 3x^2$ from $x = 0$ to $x = 2$:

j	1	2	3	4
x_j	1/4	3/4	5/4	7/4
$f(x_j)$	.108	1.69	4.69	9.19
Area of Rectangle j	.05	.845	2.35	4.60

$A \approx 7.85$

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 2$ (4 rectangles, midpoint rule): 7.85

If we continued to take more and more rectangles, can you guess what our area estimates would get closer and closer to? 8

Now, let's use the same procedure to approximate the area under $y = 3x^2$ from $x = 0$ to $x = 4$ using 2 rectangles:

j	1	2
x_j	1	3
$f(x_j)$	3	27
Area of Rectangle j	6	54

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 4$ (2 rectangles, midpoint rule): $\boxed{60}$

Now use 4 rectangles to approximate the area under $y = 3x^2$ from $x = 0$ to $x = 4$:

j	1	2	3	4
x_j	1/2	3/2	5/2	7/2
$f(x_j)$	.75	6.75	18.75	36.75
Area of Rectangle j	.75	6.75	18.75	36.75

Riemann sum for $y = 3x^2$ from $x = 0$ to $x = 4$ (4 rectangles, midpoint rule): $\boxed{63}$

If we continued to take more and more rectangles, can you guess what our area estimates would get closer and closer to? 64

Based on the approximations you got from these Riemann sums, what would you guess that the area is under the curve $y = 3x^2$ from $x = 0$ to $x = \ldots$

0	0
1	1
2	8
4	64
5	125
a	a^3

Definite Integral of $f(x)$ from a to b

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta \cancel{x} f(x_n)$$

where $a \leq x_1 < x_2 < \cdots < x_{n-1} < x_n \leq b$ and $\Delta x = x_{j+1} - x_j$. In plain English, we would say that the definite integral is the limit of Riemann Sums as the width of the rectangles gets smaller and smaller.

We now have a way to compute the area under a curve: approximate the region by rectangles, and as you increase the number of rectangles the Riemann sums give us an approximation of the definite integral. **Problem: This is horribly complicated**. However, if we can find an *antiderivative* of $f(x)$, the Fundamental Theorem of Calculus gives us a very simple way of computing the definite integral

Fundamental Theorem of Calculus. If $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Examples

(a) $\int_0^a 2xdx = F(a) - F(0) = a^2 - 0 = \boxed{a^2}$

$f(x) = 2x$

$F(x) = x^2$

(b) $\int_0^a 3x^2dx = F(a) - F(0) = a^3 - 0^3 = \boxed{a^3}$

$f(x) = 3x^2$

$F(x) = x^3$

(c) $\int_{-2}^{3} x\,dx$ $= F(3) - F(-2) = \frac{1}{2}3^2 - \frac{1}{2}(-2)^2$

$f(x) = x$

$F(x) = \frac{1}{2}x^2$

$= \frac{9}{2} - \frac{4}{2} = \boxed{5/2}$

(d) $\int_{-1}^{1} e^{2x}\,dx$ $= \frac{e^{2\cdot 1}}{2} - \frac{e^{2\cdot(-1)}}{2} = \boxed{\frac{e^{2} - e^{-2}}{2} \approx 3.63}$

$f(x) = e^{2x}$

$F(x) = \frac{e^{2x}}{2}$

(e) $\int_{1}^{4} \left(\frac{2}{5}x + \frac{2}{7}\right) dx$ $= F(4) - F(1) = \frac{4^2}{5} + \frac{2}{7}(4) - \frac{1}{5} - \frac{2}{7}$

$f(x) = \frac{2}{5}x + \frac{2}{7}$

$F(x) = \frac{x^2}{5} + \frac{2}{7}x$

$= \frac{16}{5} + \frac{8}{7} - \frac{1}{5} - \frac{2}{7}$

$= \frac{15}{5} + \frac{6}{7} = 3 + 6/7 = \boxed{\frac{27}{7}}$

(f) $\int_{-1}^{2} (x^4 + 2x^2 - 8)\,dx$ $= \frac{2^5}{5} + \frac{2\cdot 8}{3} - 8\cdot 2 - \left(\frac{(-1)^5}{5} + \frac{2(-1)^3}{3} - 8(-1)\right)$

$f(x) = x^4 + 2x^2 - 8$

$F(x) = \frac{x^5}{5} + \frac{2x^3}{3} - 8x$

$= \frac{32}{5} + \frac{\cancel{24}\ 16}{3} - 16 - \left(\frac{-1}{5} - \frac{2}{3} + 8\right)$

$= \frac{33}{5} + \frac{18}{3} - 24$

$= \frac{3\cdot 33 + 5\cdot 18}{15} - 24 = 12.6 - 24$

$= \boxed{-11.4}$

(g) $\int_1^5 \left(\frac{x}{5}+\sqrt{x}\right)dx$

$f(x) = \frac{x}{5} + x^{1/2}$

$F(x) = \frac{x^2}{10} + \frac{2}{3}x^{3/2}$

$= \frac{5^2}{10} + \frac{2}{3}5^{3/2} - \frac{1^2}{10} - \frac{2}{3}1^{3/2}$

$= 2.5 + \frac{2}{3}5^{3/2} - \frac{1}{10} - \frac{2}{3}$

≈ 9.19

(h) $\int_{-1}^1 \frac{e^x+e^{-x}}{2}dx$

$f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$

$F(x) = \frac{e^x}{2} + \frac{e^{-x}}{-2}$

$= \frac{e^x - e^{-x}}{2}$

$= \frac{e^1 - e^{-1}}{2} - \frac{e^{-1} - e^1}{2}$

$= \frac{e^1}{2} - \frac{e^{-1}}{2} - \frac{e^{-1}}{2} + \frac{e^1}{2}$

$= e^1 - e^{-1} = \boxed{e - \frac{1}{e}}$

(i) $\int_0^2 \left(e^{.5x} - 7 + 3x^{1/3}\right)dx$

$f(x) = e^{.5x} - 7 + 3x^{1/3}$

$F(x) = \frac{e^{.5x}}{.5} - 7x + 3\frac{x^{4/3}}{4/3}$

$= \frac{e^{.5x}}{.5} - 7x + \frac{9}{4}x^{4/3}$

$= \frac{e^1}{.5} - 7\cdot 2 + \frac{9}{4}2^{4/3}$

$- \left(\frac{e^0}{.5} - 7\cdot 0 + \frac{9}{4}0^{4/3}\right)$

$= \frac{e^1}{.5} - 14 + \frac{9}{4}2^{4/3} - \frac{1}{.5}$

$\boxed{\approx -4.89}$