Derivatives of Exponential Functions

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How can we find the derivative of $f(x) = 2^x$? To start, we can just use the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

To start, let's find $f'(x)$, where $f(x) = 2^x$ by filling in the empty spaces in the table below:

h	$f(0+h) - f(0)$	$\frac{f(0+h) - f(0)}{h}$
1	1	1
.01	.006956	.695
.0001		.693

h	$f(1+h) - f(1)$	$\frac{f(1+h) - f(1)}{h}$
1		2
.01		1.39
.0001		1.386

Using the tables above, estimate $f'(0)$ and $f'(1)$:

$f'(0) \approx .69$ $\qquad$ $f'(1) \approx 1.38$

Repeat this process for $f(x) = 3^x$:

h	$f(0+h) - f(0)$	$\frac{f(0+h) - f(0)}{h}$
1		2
.01		1.104
.0001		1.09

h	$f(1+h) - f(1)$	$\frac{f(1+h) - f(1)}{h}$
1		~~8~~ 6
.01		3.31
.0001		3.29

Using the tables above, estimate $f'(0)$ and $f'(1)$:

$f'(0) \approx 1.09$ $\qquad$ $f'(1) \approx 3.29$

Repeat this process for $f(x) = e^x$:

h	$f(0+h) - f(0)$	$\frac{f(0+h) - f(0)}{h}$
1		1.71
.01		1.005
.0001		1.00005

h	$f(1+h) - f(1)$	$\frac{f(1+h) - f(1)}{h}$
1		4.67
.01		2.73
.0001		2.718

Using the tables above, estimate $f'(0)$ and $f'(1)$:

$f'(0) \approx$ 1 $\qquad$ $f'(1) \approx$ 2.718

Derivative of e^x:

$$\frac{d}{dx}e^x = e^x$$

Practice Problems

Find the derivative of the following:

1. $f(x) = xe^x$

$$xe^x + (1)e^x = xe^x + e^x = (x+1)e^x$$

2. $f(x) = \dfrac{e^x + 1}{x}$

$$= \frac{xe^x - (e^x+1)(1)}{x^2}$$

3. $f(x) = e^x(\sqrt{x^2+1})$

$$e^x \frac{d}{dx}(x^2+1)^{1/2} + e^x\sqrt{x^2+1}$$

$$e^x \tfrac{1}{2}(x^2+1)^{-1/2}(2x) + e^x\sqrt{x^2+1}$$

4. The function $f(x) = x - e^x$ has one critical point. Find this point and determine if it is a maximum or a minimum.

$f'(x) = 1 - e^x$

$0 = 1 - e^x$

$e^x = 1$

$x = 0 \leftarrow$ max at $x = 0$

$f''(x) = -e^x$

$f''(0) = -e^0 = -1 < 0$

Chain Rule with Exponential Functions

Now we know the derivative of e^x. How do we differentiate e^{5x}? How about e^{x^2}?

Chain Rule for Exponential Functions:

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

More Practice

Find the derivatives of the following:

1. $y = e^{5x}$

$y' = 5e^{5x}$

2. $g(t) = e^{t^2}$

$g'(t) = 2te^{t^2}$

$y = e^{t^3}$

$y' = e^{t^3} 3t^2$

3. $h(x) = xe^{5x^2}$

$h'(x) = x\left(10xe^{5x^2}\right) + (1)e^{5x^2}$

4. $y = e^{\sqrt{2x^3+x}}$

$\frac{d}{dx}\sqrt{2x^3+x} = \frac{d}{dx}(2x^3+x)^{1/2} = \frac{1}{2}(2x^3+x)^{-1/2}(6x^2+1)$

$y' = \frac{1}{2}(2x^3+x)^{-1/2}(6x^2+1)e^{\sqrt{2x^3+x}}$

Extra Derivative Practice

1. $y = \left(e^{2x}\right)^4$
2. $y = \left(e^{2}x\right)^4$
3. $f(x) = e^{3x^2+5x}$
4. $f(x) = \dfrac{2}{1+e^{5x}}$
5. $g(t) = \sqrt{t^2+4}\,e^{t^2}$
6. $y = \dfrac{e^x}{x^2+4}$
7. $f(x) = 5x^3 + 6x - 2e^{-3x}$
8. $f(x) = (4-5e^x)^3$
9. $g(x) = x^2e^x - 2xe^x + 2e^x$
10. $y = (x+1)^3e^{4x}$
11. $f(x) = x^4e^x$
12. $y = e^{x^2} - x^{e^2}$

① $y = \left(e^{2x}\right)^4 = e^{8x}$

$\boxed{y' = 8e^{8x}}$

② $y = (e^2x)^4 = e^8x^4$

$\boxed{y' = 4e^8x^3}$

③ $\boxed{f'(x) = (6x+5)e^{3x^2+5x}}$

④ $f(x) = 2(1+e^{5x})^{-1}$

$f'(x) = -2(1+e^{5x})^{-2}(5e^{5x})$

$\boxed{f'(x) = \dfrac{-10e^{5x}}{(1+e^{5x})^2}}$

⑤ $\boxed{g'(t) = \dfrac{t}{\sqrt{t^2+4}}e^{t^2} + 2t\sqrt{t^2+4}\,e^{t^2}}$

⑥ $y' = \dfrac{(x^2+4)e^x - e^x(2x)}{(x^2+4)^2}$

$\boxed{y' = \dfrac{x^2-2x+4}{(x^2+4)^2}e^x}$

⑦ $\boxed{f'(x) = 15x^2 + 6 + 6e^{-3x}}$

⑧ $f'(x) = 3(4-5e^x)^2(-5e^x)$

$\boxed{f'(x) = -15e^x(4-5e^x)^2}$

⑨ $g'(x) = \cancel{2xe^x} + x^2e^x$

$- \cancel{2e^x} - \cancel{2xe^x} + \cancel{2e^x}$

$\boxed{g'(x) = x^2e^x}$

(10) $\boxed{y' = 3(x+1)^2 e^{4x} + 4(x+1)^3 e^{4x}}$

(11) $\boxed{f'(x) = 4x^3 e^{x} + x^4 e^{x}}$

(12) $\boxed{y' = 2x e^{x^2} - e^2 x^{e^2 - 1}}$